HONOURS PROJECT

THEORIZING THE EXISTENCE OF MAGNETIC MONOPOLES

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Abstract

This is work under progress

1 Magnetic Monopoles

We know of the existence of two kinds of electric charges: positive and negative. Similarly, we observe the existence of two types of magnetic poles: the north and south poles. However, these poles appear to always occur in pairs and for instance an attempt in cutting a bar magnet into two always results in two dipoles with parallel orientations instead of a single pole. There are as of present, no known occurrences nor processes to obtain independent magnetic poles, that is, magnetic monopoles or magnetic charges. However, the existence magnetic monopoles or magnetic charges is supported by the Superstring Theory and the Grand Unified Theory and should they exist, they would explain why electric charge is quantized and would also make the Maxwell's equations symmetric.

2 Maxwell's Equations

The Maxwell's equations, which describe the influence of Electric and Magnetic fields on each other, are given by:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss's Law)} \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(Gauss's Law [Magnetism])} \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampere-Maxwell's Law)} \tag{2.4}$$

2.1 Gauss's Law derivation

From Coulomb's Law we know that the electric field due to stationary electric charge q_e at a distance r is given by:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_e \hat{\mathbf{r}}}{r^2} \tag{2.5}$$

The net electric field at r infinitesimal charges at each point represented by $\mathbf{s}$ is given by:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho_e(s)(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3} d^3 s \tag{2.6}$$

Now, evaluating the gradient on both sides, we have:

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \nabla \cdot \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho_e(s)(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3} d^3 s = \frac{1}{4\pi\epsilon_0} \iiint_V \nabla \cdot \frac{\rho_e(s)(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3} d^3 s \tag{2.7}$$

Using the property of the three-dimensional Dirac Delta function which states that $\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta^3(\mathbf{r})$ we have:

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0} \iiint_V \rho_e(s) \delta^3(\mathbf{r}-\mathbf{s}) d^3 s \tag{2.8}$$

The sifting property of the integral of a time-delayed Dirac Delta function f by a T is given by:

$$\int f(t)\delta(t-T)dt = f(T) \tag{2.9}$$

Applying this sifting property we have :

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho_e(r)}{\epsilon_0} \tag{2.10}$$

The Gauss Law signifies that the net flux of the electric field linked through a closed surface is equal to the total charge enclosed by the closed surface.The equation also supports the existence of positive and negative electric charges as the positive charges act as the source of flux of the electric field and the negative charges act as the sink of flux of the electric field.

2.2 Gauss's Law for Magnetism derivation

We know that the movement of electric charges induces magnetic fields. Let us make the assumption that all magnetic fields are a result of moving electric charges. By Biot-Savart's Law we know that the magnetic field due to a current I through a length dl at a distance r and a distance r' to the infinitesimal element dl is given by:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{dl \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{2.11}$$

Evaluating the gradient on both sides we obtain:

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \nabla \cdot \frac{dl \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{2.12}$$

$$= \frac{|\mathbf{r} - \mathbf{r}'|^3 \nabla \cdot [dl \times (\mathbf{r} - \mathbf{r}')] - \nabla \left[|\mathbf{r} - \mathbf{r}'|^3 \right] \cdot [dl \times (\mathbf{r} - \mathbf{r}')]}{|\mathbf{r} - \mathbf{r}'|^6} \tag{2.13}$$

The Gauss Law for Magnetism suggests that the flux of magnetic induction across any closed surface is always zero. This supports experimental results that show that the number of magnetic field lines entering any arbitrary closed surface is equal to number of magnetic field lines leaving it. Thus, by suggesting that magnetic flux lines are always closed loops implies that there are no sink or sources in the case of magnetic flux. It therefore, does not allow the existence of magnetic monopoles and establishes that magnetic poles always occur in pairs as dipoles.

2.3 Maxwell's Third equation derivation

From Faraday's law of electromagnetic induction, we know that the induced electromotive force(EMF) in a loop is equal to the rate of change of magnetic flux linkage. This is otherwise written as:

$$\varepsilon = -\frac{\mathrm{d}\phi_B}{\mathrm{d}t} \tag{2.14}$$

We also know that the flux is given by:

$$\phi_B = \int_S \mathbf{B} \cdot \mathbf{ds} \tag{2.15}$$

From equations 2.14 and 2.15 we obtain:

$$\varepsilon = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathbf{ds} \tag{2.16}$$

$$= -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{ds} \tag{2.17}$$

Since the EMF by definition is the work done in carrying a unit charge around a closed loop in an electric field the EMF can also be written as:

$$\varepsilon = \int_C \mathbf{E} dl \tag{2.18}$$

From equations 2.17 and 2.18 we have:

$$\int_C \mathbf{E} dl = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{ds} \tag{2.19}$$

From Stokes Theorem we know that:

$$\int_C \mathbf{E} dl = \int_S (\nabla \times \mathbf{E}) ds \tag{2.20}$$

Applying Stokes Theorem to the Left-hand side of equation 2.19:

$$\int_S (\nabla \times \mathbf{E}) ds = -\int_S \frac{\partial \mathbf{B}}{\partial t} ds \tag{2.21}$$

$$\Rightarrow \int_S \left[(\nabla \times \mathbf{E}) + \frac{\partial \mathbf{B}}{\partial t} \right] \mathbf{ds} = 0 \tag{2.22}$$

Since the solution to the above equation must be true for all surfaces we have:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.23}$$

2.4 Ampère's law with Maxwell's addition derivation

2.5 Maxwell's Equations in Free Space

In free space, the charge density, $\rho = 0$ and consequently current density, $\mathbf{J} = \rho \mathbf{E} = 0$. Additionally $\epsilon = \epsilon_0$ and $\mu = \mu_0$. Substituting these values in equations 2.1-2.4 results in the following Maxwells equations in free space:

$$\nabla.\mathbf{E} = 0 \qquad \nabla.\mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.24}$$

There is a striking symmetry that can be noted from above in the Maxwell's equations in free space where on replacement of $\mathbf{E}$ with $\mathbf{B}$ and $\mathbf{B}$ with $-\epsilon_0\mu_0\mathbf{E}$ the equations remain unchanged.

2.6 Restoring Dual Symmetry of Maxwell's Equations

Now, in order to retain this symmetry in the general forms of the Maxwell's Equations let us assume that magnetic charges exist. We define ρ_m to be the charge density of magnetic charges and ρ_e to the charge density of electric charges. We also define the magnetic current density and electric current density to be given by $\nabla.\mathbf{J}_m = -\frac{\partial \rho_m}{\partial t}$ and $\nabla.\mathbf{J}_e = -\frac{\partial \rho_e}{\partial t}$ respectively. Incorporating these and applying the Law of Universal Magnetism, the modified Maxwell's equations given by:

$$\nabla.\mathbf{E} = \frac{\rho_e}{\epsilon_0} \tag{2.25}$$

$$\nabla.\mathbf{B} = \mu_0 \rho_m \tag{2.26}$$

$$\nabla \times \mathbf{E} = -\mu_0 \mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t} \tag{2.27}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_e + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.28}$$

It can be noted how the above equations are invariant under the exchange of electric and magnetic field components, thus showing that the existence of magnetic monopoles would restore the dual symmetry noted in free space.

2.7 Deriving Gauss's Law for Magnetism with Magnetic Monopoles

In the derivation of Gauss's Law for Magnetism earlier, we made the assumption that the only Magnetic field is due to the field induced by moving electric charges. However, taking the possibility of the existence of magnetic monopoles or magnetic charges gives us a new potential source of magnetic fields. The Universal Law of Magnetism[1] states that the magnetic field due to a magnetic charge q_m at a distance r is given by:

$$\mathbf{B} = \frac{\mu q_m \hat{\mathbf{r}}}{4\pi r^2} \tag{2.29}$$

The net magnetic field at r from infinitesimal charges at each point $\mathbf{s}$ of a charge distribution is given by:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu}{4\pi} \iiint_V \frac{\rho_m(s)(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3} d^3 s \tag{2.30}$$

This is analogous to the derivation of the Gauss's Law from Section 2.1. Now, evaluating the gradient on both sides, we have:

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \nabla \cdot \frac{\mu}{4\pi} \iiint_V \frac{\rho_m(s)(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3} d^3 s = \frac{\mu}{4\pi} \iiint_V \nabla \cdot \frac{\rho_m(s)(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3} d^3 s \tag{2.31}$$

Using the property of the three-dimensional Dirac Delta function which states that $\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta^3(\mathbf{r})$ we have:

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \mu \iiint_V \rho_m(s)\delta^3(\mathbf{r}-\mathbf{s}) d^3 s \tag{2.32}$$

The sifting property of the integral of a time-delayed Dirac Delta function f by a T is given by:

$$\int f(t)\delta(t-T)dt = f(T) \tag{2.33}$$

Applying this sifting property we have :

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \mu \rho_m(r) \tag{2.34}$$

3 Dirac's quantization of electric charges

3.1 Aharanov-Bohm Effect

3.2 Heuristic Approach using Angular Momentum of due to an electric charge and magentic charge

4 Groundwork theory

4.1 Mathematics

4.1.1 Dirac-delta Function

4.2 Lagrangian Mechanics

4.3 Relativity

4.3.1 Inertial Reference Frame

An inertial reference frame is a reference frame in which Newton's Laws hold. It is characterised as an non-accelerated reference frame. This means that if there exists a body with no net external force acting on it, the motion of the body will be non-accelerated as observed from this reference frame. All motion is said to be relative, and hence, there exists no absolute reference frame in this universe.

4.3.2 Newtonian Relativity

4.3.3 Special Relativity

Newtonian mechanics proposes no theoretical limit to the velocity of a particle. However, experimental evidence shows that predictions based on Newtonian Mechanics do not hold ground for speeds that approach the speed of light. In order to understand this and also gain further insight into the nature of electromagnetism, Einstein revisited the foundations of Newtonian Mechanics, namely the notion of space/distance and time.

4.4 Electrodynamics

4.5 Quantum Mechanics

References

[1] "Law of Universal Magnetism, F = k × H". In: 04 (Jan. 2018), pp. 471–484. DOI: 10.4236/jhepgc.2018.43025.