PRMO 2017

1. How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3? 28

2. Suppose a, b are positive real numbers such that $a\sqrt{a} + b\sqrt{b} = 183$, $a\sqrt{b} + b\sqrt{a} = 182$. Find $\frac{9}{5}(a+b)$.

3. A contractor has two teams of workers: team A and team B. Team A can complete a job in 12 days and team B can do the same job in 36 days. Team A starts working on the job and team B joins team A after four days. The team A withdraws after two more days. For how many more days should team B work to complete the job? 02

4. Let a, b be integers such that all the roots of the equation $(x^2+ax+20)(x^2+17x+b)=0$ are negative integers. What is the smallest possible value of $a+b$?

5. Let u, v, w be real numbers in geometric progression such that $u > v > w$. Suppose $u^{40} = v^n = w^{60}$. Find the value of n.

6. Let the sum $\sum_{n=1}^{9} \frac{1}{n(n+1)(n+2)}$ written in its lowest terms be $\frac{p}{q}$. Find the value of $q-p$. 83

7. Find the number of positive integers n, such that $\sqrt{n} + \sqrt{n+1} < 11$. 05

8 A pen costs ₹11 and a notebook costs ₹13. Find the number of ways in which a person can spend exactly ₹1000 to buy pens and notebooks.

9. There are five cities A, B, C, D, E on a certain island. Each city is connected to every other city by road. In how many ways can a person starting from city A come back to A after visiting some cities without visiting a city more than once and without taking the same road more than once? (The order in which he visits the cities also matters: e.g., the routes $A \to B \to C \to A$ and $A \to C \to B \to A$ are different.) 24

10. There are eight rooms on the first floor of a hotel, with four rooms on each side of the corridor, symmetrically situated (that is each room is exactly opposite to one other room). Four guests have to be accommodated in four of the eight rooms (that is, one in each) such that no two guests are in adjacent rooms or in opposite rooms. In how many ways can the guests be accommodated? 02

11. Let $f(x) = \sin\frac{x}{3} + \cos\frac{3x}{10}$ for all real x. Find the least natural number n such that $f(n\pi + x) = f(x)$ for all real x.

12. In a class, the total numbers of boys and girls are in the ratio $4:3$. On one day it was found that 8 boys and 14 girls were absent from the class, and that the number of boys was the square of the number of girls. What is the total number of students in the class? 42

13. In a rectangle $ABCD$, E is the midpoint of AB; F is a point on AC such that BF is perpendicular to AC; and FE perpendicular to BD. Suppose $BC = 8\sqrt{3}$. Find AB.

14. Suppose x is a positive real number such that $\{x\}, [x]$ and x are in a geometric progression. Find the least positive integer n such that $x^n > 100$. (Here $[x]$ denotes the integer part of x and $\{x\} = x - [x]$.)

15. Integers $1, 2, 3, \ldots, n$, where $n > 2$, are written on a board. Two numbers m, k such

$x^2 + 2xy + 4y^2 + 2(2y)$

that $1 < m < n$, $1 < k < n$ are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers?

16. Five distinct 2-digit numbers are in a geometric progression. Find the middle term.

17. Suppose the altitudes of a triangle are 10, 12 and 15. What is its semi-perimeter?

18. If the real numbers x, y, z are such that $x^2 + 4y^2 + 16z^2 = 48$ and $2xy + 8yz + 4zx = 24$. 48 what is the value of $x^2 + y^2 + z^2$? $xy + 4yz + 2zx = 24$

19. Suppose 1, 2, 3 are the roots of the equation $x^4 + ax^2 + bx = c$. Find the value of c. -14

20. What is the number of triples (a, b, c) of positive integers such that (i) $a < b < c < 10$ and (ii) $a, b, c, 10$ form the sides of a quadrilateral?

21. Find the number of ordered triples (a, b, c) of positive integers such that $abc = 108$ 54

22. Suppose in the plane 10 pairwise nonparallel lines intersect one another. What is the maximum possible number of polygons (with finite areas) that can be formed?

23. Suppose an integer x, a natural number n and a prime number p satisfy the equation $7x^2 - 44x + 12 = p^n$. Find the largest value of p. 02

24. Let P be an interior point of a triangle ABC whose sidelengths are 26, 65, 78. The line through P parallel to BC meets AB in K and AC in L. The line through P parallel to CA meets BC in M and BA in N. The line through P parallel to AB meets CA in S and CB in T. If KL, MN, ST are of equal lengths, find this common length.

25. Let $ABCD$ be a rectangle and let E and F be points on CD and BC respectively such that area$(ADE) = 16$, area$(CEF) = 9$ and area$(ABF) = 25$. What is the area of triangle AEF?

26. Let AB and CD be two parallel chords in a circle with radius 5 such that the centre O lies between these chords. Suppose $AB = 6$, $CD = 8$. Suppose further that the area of the part of the circle lying between the chords AB and CD is $(m\pi + n)/k$, where m, n, k are positive integers with $\gcd(m, n, k) = 1$. What is the value of $m + n + k$?

27. Let Ω_1 be a circle with centre O and let AB be a diameter of Ω_1. Let P be a point on the segment OB different from O. Suppose another circle Ω_2 with centre P lies in the interior of Ω_1. Tangents are drawn from A and B to the circle Ω_2 intersecting Ω_1 again at A_1 and B_1 respectively such that A_1 and B_1 are on the opposite sides of AB. Given that $A_1B = 5$, $AB_1 = 15$ and $OP = 10$, find the radius of Ω_1.

28. Let p, q be prime numbers such that $n^{3pq} - n$ is a multiple of $3pq$ for **all** positive integers n. Find the least possible value of $p + q$.

29. For each positive integer n, consider the highest common factor h_n of the two numbers $n! + 1$ and $(n+1)!$. For $n < 100$, find the largest value of h_n.

30. Consider the areas of the four triangles obtained by drawing the diagonals AC and BD of a trapezium $ABCD$. The product of these areas, taken two at time, are computed. If among the six products so obtained, two products are 1296 and 576, determine the square root of the maximum possible area of the trapezium to the nearest integer.

44×44
176
1760
1936

$44 \pm \sqrt{}$

$(x)^2 + (2y)^2 + (4z)^2 = 48$

$\Rightarrow 4xy + 16yz + 8zx = 96$

$\Rightarrow 2(x)(2y) + 2(2y)(4z) + 2(4z)(x) = 96$

$\Rightarrow (x+y+z)^2 = 48 + 96$

$\Rightarrow (x + 2y + 4z)^2 = 144$

$\Rightarrow (x + 2y + 4z) = \pm 12$

20/08/17

107 If the sum of the ~~numbers~~ digits is divisible by 7 ~~we find~~ and if the number is < 1000 we find that the greatest ~~numb~~ multiple of 7 that can be obtained using the digits is 21 (through 993)

$\therefore$ Sum of the digits ~~of~~ of the number should be 7, 14 or 21.

However, through the divisibility test for the number being divisible by 3 is that the sum of the ~~numbers~~ digits should be divisible by 3.

Hence, the sum of ~~vv~~ the digits of the number cannot be 7 or 14, and therefore is 21.

None of the digits of the number can be 0, 1 ~~2~~ or ~~5~~ 2, as, if the digits fall within these numbers the other / (either one of) other two digits will have to be greater than 10.

If the first ~~no~~ digit is 3 $\Rightarrow$ other 2 are (9, 9) to obtain sum = 18

" " ~~second~~ " " 4 $\Rightarrow$ req.d sum - 17, possibilities for other 2 are (9, 8)

" " ~~third~~ " " 5 $\Rightarrow$ req.d sum = 16, possibilities for other 2 are (9, 8), (9, 7)

" " " " " 6 $\Rightarrow$ req.d sum = 15, possibilities for other 2 are (7, 8), (6, 9)

" " " " " 7 $\Rightarrow$ req.d sum = 14, possibilities for other 2 are (5, 9), (6, 8), (7, 7)

" " " " " 8 $\Rightarrow$ req.d sum = 13, possibilities for other 2 are (9, 4), (5, 8), (7, 6)

" " " " " 9 $\Rightarrow$ req.d sum = 12, possibilities for other 2 are (9, 3), (8, 4), (7, 5), (6, 6).

$\therefore$ (We have starting) with 3 = 1, with 4 = 2, with 5 = 3, with 6 = 4, with 7 = 5, with 8 = 6, with 9 = 7.

$\therefore$ Total $= 1+2+3 \ldots\ldots + 7 = \frac{7(7+1)}{2} = \frac{7 \times 8}{2} = 7 \times 4 = \boxed{28}$

20/08/17

2.7 Given - $a\sqrt{a} + b\sqrt{b} = 183$ - ①

$a\sqrt{b} + b\sqrt{a} = 182$ - ②

From ② $\Rightarrow \sqrt{ab}(\sqrt{a} + \sqrt{b}) = 182$ - ③

Adding ①, ② $\Rightarrow \sqrt{a}(a+b) + \sqrt{b}(a+b) = 183 + 182$

$\Rightarrow (a+b)(\sqrt{a} + \sqrt{b}) = 365$ - ④

Subtracting ② from ① $\Rightarrow \sqrt{a}(a-b) + \sqrt{b}(a-b) = 183-182$

$\Rightarrow (\sqrt{a} + \sqrt{b})(a-b) = 1$ - ⑤

Dividing eq^n ④ by ⑤ $\Rightarrow \dfrac{(a+b)(\sqrt{a}+\sqrt{b})}{(a-b)(\sqrt{a}+\sqrt{b})} = \dfrac{365}{1}$

$\Rightarrow \dfrac{(a+b)}{(a-b)} = 365$

$\Rightarrow a+b = 365a - 365b$

$\Rightarrow 366b = 364a$

$\Rightarrow 183b = 182a$

Let $\sqrt{a} = x$, $\sqrt{b} = y$

From eq① $\Rightarrow x^3 + y^3 = 183$ - ⑥

From eq② $\Rightarrow x^2y + xy^2 = 182$ - ⑦

multiplying by 3 in RHS & LHS in eq ⑦

$\Rightarrow 3x^2y + 3xy^2 = 546$ - ⑧

Adding LHS & RHS of eqns ⑥ & ⑧

$\Rightarrow x^3 + y^3 + 3x^2y + 3xy^2 = 729$

$\Rightarrow (x+y)^3 = (9)^3$

$\Rightarrow (x+y) = 9$

$\Rightarrow \sqrt{a} + \sqrt{b} = 9$ - ⑨

Substituting eqn ⑨ in eq ④

$\Rightarrow a+b = \dfrac{365}{9}$

Req.d $\dfrac{9}{5}(a+b)$

$= \dfrac{9}{5} \times \dfrac{365}{9} = \boxed{73}$

3.7 Until team B joins team A, team A has worked for 4 days and then continue to work for 2 more days with team B

=> Team has worked for 6 days.

$\because$ Team A takes 12 days to finish the job

=> In 6 days they have finished $1/2$ the job.

(assuming the portion of the job finished each day is the same)

3 1/2 min

Team B works with team A for 2 days.

$\because$ team B takes 36 days to finish the job.

=> In 2 days they have finished $2/36 = 1/18$ of the job (assuming the portion of the job finished each day is the same).

$\therefore$ The remaining part of the job $= 1-(1/2+1/18)$

$= 1 - 10/18$

$= \frac{8}{18} = \frac{4}{9}$

$\therefore$ It will take team B $\frac{4}{9} \times 36$ days to finish the job $= 4 \times 4 = \boxed{16}$ days //

4.7 Let the roots of $x^2+ax+20=0$ be α, β

=> $\alpha+\beta = -a$

$\alpha\beta = 20$

4 min

Possibilities for $\alpha, \beta = (-20,-1), (-10,-2), (-5,-4)$ ($\because$ roots are given to be negative)

=> " " $\underset{(\alpha+\beta)}{-a} = +(-21), +(-12), +(-9)$

=> " " $a = 21, 12, 9$.

Let the roots of $x^2+17x+b=0$ be γ, δ

=> $\gamma+\delta = -17$

$\gamma\delta = b$

Possibilities for $\gamma, \delta = (-16,-1), (-15,-2), (-14,-3), (-13,-4)$ ..... $(-9,-8)$

=> Possibilities for $\underset{(b)}{\gamma\delta} = 16, 30, 42, 52 \ldots\ldots 72$

But the smallest value of $a = 9$

and the smallest value of $b = 16$

$\therefore$ The smallest value of $a + b = 9 + 16$

$= \boxed{25}$

21/08/17

5.> u, v, w are in G.P.

$\Rightarrow \frac{v}{u} = \frac{w}{v} =$ constant (common ratio)

Let constant $= r$

$\Rightarrow v = ur$ -- (1)

$w = vr$ -- (2)

$\Rightarrow w = ur^2$ -- (3)

Given $u^{40} = w^{60}$ -- (4)

$\Rightarrow (r^2u)^{60} = w^{60}$ (From eq (3))

$\Rightarrow r^{120} * u^{60} = w^{60}$

$\Rightarrow r^{120} u^{60} = u^{40}$ (From eq (4))

$\Rightarrow r^{120} = u^{-20}$

$\Rightarrow r^6 = u^{-1}$

$\Rightarrow r = \frac{1}{\sqrt[6]{u}} = u^{-1/6}$ -- (5)

Substituting (5) in eq (1) & (2)

$v = u \times u^{-1/6} = u^{5/6}$

$w = u \times u^{-2/6} = u^{2/3}$

Given $u^{40} = v^n$

$\Rightarrow u^{40} = (u^{5/6})^n$

$= u^{5n/6}$

$\Rightarrow \cancel{40}^{8} = \frac{\cancel{5}n}{6}$

$\Rightarrow \boxed{48 = n}$

6.y $$\sum_{n=1}^{9} \frac{1}{n(n+1)(n+2}$$

$$= \sum_{n=1}^{9} \frac{1}{2}\left(\frac{2}{n(n+1)(n+2)}\right)$$

$$= \frac{1}{2} \sum_{n=1}^{9} \left(\frac{(n+2)-n}{n(n+1)(n+2)}\right)$$

$$= \frac{1}{2} \sum_{n=1}^{9} \left(\frac{\cancel{n+2}}{n(n+1)\cancel{(n+2)}}\right) - \left(\frac{\cancel{n}}{\cancel{n}(n+1)(n+2}\right)$$

$$= \frac{1}{2} \sum_{n=1}^{9} \left(\frac{1}{n(n+1}\right) - \left(\frac{1}{(n+1)(n+2)}\right)$$

$$= \frac{1}{2}\left(\left(\frac{1}{2} - \frac{1}{2\times3}\right) + \left(\frac{1}{2\times3} - \frac{1}{3\times4}\right) + \left(\frac{1}{3\times4} - \frac{1}{4\times5}\right) \cdots \left(\frac{1}{9\times10} - \frac{1}{10\times11}\right)\right)$$

Since all consecutive terms excluding the first and the last will cancel

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{10\times11}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{110}\right)$$

$$= \frac{1}{2}\left(\frac{110-2}{220}\right)$$

$$= \frac{1}{2}\left(\frac{\cancel{108}^{\cancel{54}\,27}}{\cancel{220}_{\cancel{110}\,55}}\right)$$

$$= \frac{27}{110}$$

$\therefore \frac{p}{q} = \frac{27}{110}$ & $p = 27$, $q = 110$

$\therefore q - p = 110 - 27 = \boxed{83}$

7.) $\sqrt{n} + \sqrt{n+1} < 11 \quad -①$

$\Rightarrow \sqrt{n+1} < 11 - \sqrt{n}$

Squaring both sides

$\Rightarrow \cancel{n}+1 < 121 - 22\sqrt{n} + \cancel{n}$

$\Rightarrow 22\sqrt{n} < 120$

$\Rightarrow 11\sqrt{n} < 60$

$\Rightarrow \sqrt{n} < \frac{60}{11}$

$\Rightarrow n < \frac{3600}{121}$

$\Rightarrow n < 29.7\ldots$

$\Rightarrow n \leq 29$

$\therefore$ The number of positive natural numbers satisfying $\sqrt{n} + \sqrt{n+1} < 11$ is $\boxed{29}$.//

8.) Let the number of pens = x

Let the number of notebooks = y.

$\Rightarrow 11x + 13y = 1000$

$\Rightarrow 11x \cancel{+13} = 1001 - 1 - 13y$

$\Rightarrow \cancel{11x = 13(77-y) - 1} \quad 11x = 1001 - 11y - (2y+1)$

$\Rightarrow \cancel{13x - 2x = 13(77-y) - 1} \quad 11x = 11(91-y) - (2y+1)$

$\Rightarrow \cancel{13(x - 77 - y) = 2x - 1} \quad 2y+1$ is divisible by 11.

Let $2y+1 = 11(2k+1)$

$\Rightarrow x = 91 - y - 2k - 1$

$= 90 - y - 2k$

21/08/12

9. Starting from city A, 4 roads can be chosen and on reaching ~~[illegible]~~ the first city one has 3 roads that can be chosen and on reaching second city one has 2 roads that can be chosen and finally only 1 road. This is true because we are not permitted to use one road twice.

1½ mins

$\therefore$ The total number of ways in which one can start from A goto ~~all these~~ some cities and come back without using a road twice

$= {}^4P_4 + {}^4P_3 + {}^4P_2$

$= 4! + 4! + 4\times3$

$= 24 + 24 + 12$

$= \boxed{60}$

10.

O X O X

X O X O

1½ mins

There are 2 ways of selecting the 4 rooms as once you fix the first room there is only one way of selecting the next room and the rooms that follow.

And the number of ways of arranging the 4 guests in ~~way of~~ the chosen 4 rooms = 4!

$\therefore$ The total number of ways in which the guests can be accomodated $= 2\times4!$

$= 2\times4\times3\times2\times1$

$= 2\times24$

$= \boxed{48}$

11.

21/08/17

12. Let the number of boys in the class $= 4x$

" " " " girls " " " $= 3x$

$\Rightarrow$ The total number of students in the class $= 7x$

One day,

no. of boys in class $= 4x - 8$

no. of girls in class $= 3x - 14$

on this day it was found that

no. of boys $=$ (no. of girls)^2

$\Rightarrow 4x - 8 = (3x - 14)^2$

$\Rightarrow 4x - 8 = 9x^2 - 84x + 196$

$\Rightarrow 9x^2 - 88x + 204 = 0$

$\Rightarrow x = \dfrac{-(-88) \pm \sqrt{7744 - 7344}}{18}$

$= \dfrac{88 \pm \sqrt{400}}{18}$

$= \dfrac{88 \pm 20}{18}$

$= \dfrac{54}{9}, \dfrac{34}{9}$

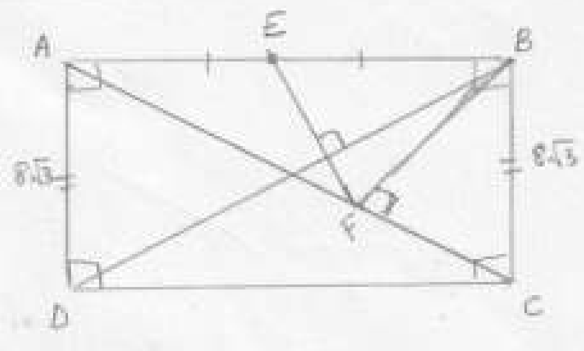
$= 6, \dfrac{34}{9}$

However, since x has to be a positive integer $x = 6$.

$\Rightarrow$ The total number of students in the class

$= 7x = 7(6) = \boxed{42}$

13.



14.) $\{x\}, [x], x$ are in G.P.

$\Rightarrow \frac{x}{[x]} = \frac{[x]}{\{x\}} =$ constant (the common ratio of the G.P)

• Let this constant $= r$

$\Rightarrow \frac{x}{[x]} = \frac{[x]}{\{x\}} = r$

$\Rightarrow \frac{x}{[x]} = \frac{[x]}{x-[x]} = r$

or $x^2 - [x]x = [x]^2$

$\Rightarrow x^2 - [x]x - [x]^2 = 0$

$\Rightarrow \frac{[x] \pm \sqrt{[x]^2 + 4[x]^2}}{2} = x$

$\Rightarrow [x]\left(\frac{1 \pm \sqrt{5}}{2}\right) = x$

$\Rightarrow \frac{1 \pm \sqrt{5}}{2} = \frac{x}{[x]}$

$\Rightarrow \frac{1 \pm \sqrt{5}}{2} = r$

However since $x, [x]$ are positive and $\sqrt{5} > 1$

$\Rightarrow r = \frac{1+\sqrt{5}}{2}$

$x = [x] r$

$\frac{x-[x]}{[x]} = \frac{x}{[x]} - 1$

$\Rightarrow \frac{x}{[x]} = \frac{1+\sqrt{5}}{2} - \frac{2}{2}$

$= \frac{\sqrt{5}-1}{2}$

But $\frac{\sqrt{5}-1}{2} = \frac{1}{r}$

∴ By observing patterns and trial and error we find $[x] = 1$, $x = \sqrt{5}+1$

We require the smallest number n such that $x^n > 100$

But $x = \frac{\sqrt{5}+1}{2}$

$\therefore$ Req.d :- smallest power of n such that

$$\left(\frac{\sqrt{5}+1}{2}\right)^n > 100$$

$$\frac{\sqrt{5}+1}{2} < \sqrt{3}$$

$\Rightarrow$ And $(\sqrt{3})^8 = 81 < 100$

$$\therefore \left(\frac{\sqrt{5}+1}{2}\right)^8 < (\sqrt{3})^8 < 100$$

$$\therefore n > 8$$

$$\left(\frac{\sqrt{5}+1}{2}\right)^n > 100$$

$$\Rightarrow \left(\frac{\sqrt{5}+1}{2}\right)^{\frac{n}{2}} > 10$$

Let $n/2 = m$

$\Rightarrow n = 2m$

$$\left(\frac{\sqrt{5}+1}{2}\right)^5 = \frac{(1+\sqrt{5})^2(1+\sqrt{5})^2}{16}\left(\frac{1+\sqrt{5}}{2}\right)$$

$$= \frac{(6+2\sqrt{5})(6+2\sqrt{5})}{16}\left(\frac{1+\sqrt{5}}{2}\right)$$

$$= \left(\frac{36+24\sqrt{5}+20}{16}\right)\left(\frac{1+\sqrt{5}}{2}\right)$$

$$= \frac{56+24\sqrt{5}\,(1+\sqrt{5})}{32}$$

$$= \frac{80\sqrt{5}+176}{32}$$ $\Rightarrow$ Numerator is $>$ $10 \times$ Denominator

Reason

$\sqrt{5} > 2$

$\Rightarrow 80\sqrt{5} > 160$

$\Rightarrow 80\sqrt{5}+176 > 160+176$

$\therefore m = 5$

$\Rightarrow n = 2m = \boxed{10}$

15.→

$$\frac{(1+2+3 \cdots\cdots n) - (m+k)}{(n-2)} = 17$$

$$\Rightarrow \frac{n(n+1)}{2} - (m+k) = 17(n-2)$$

$$\Rightarrow m+k = \frac{n(n+1)}{2} - 17(n-2)$$

$$= \frac{n^2+n - 34(n-2)}{2}$$

$$= \frac{n^2 \overset{-33n}{\cancel{+35n}} \cancel{+34} + 68}{2}$$

$$= \frac{n^2 - 33n + 68}{2}$$

16.)

Let the 5 numbers be

ar, ar, ar^2, ar^3, ar^4

$10 \leq a \leq 99$, $r \geq 1$

$\Rightarrow ar^4 < 100$

$\Rightarrow r^4 \leq \frac{100}{a}$

But since $a \geq 10$

$\Rightarrow \frac{100}{a} \leq 10$

$\therefore r^4 \leq 10$

$\Rightarrow r \leq 2$

17.) $h_1 = 10,\ h_2 = 12,\ h_3 = 15$

$ar(\triangle ABC) = \frac{1}{2} \times 10 \times a = \frac{1}{2} \times 12 \times b = \frac{1}{2} \times 15 \times c$

$\Rightarrow 2\,ar(\triangle ABC) = 10a = 12b = 15c$

$\therefore 2\,ar(\triangle ABC)$ is divisible by 10, 12, 15.

$\Rightarrow 2\,ar(\triangle ABC) = 60\alpha$

$\Rightarrow ar(\triangle ABC) = 30\alpha$

$\Rightarrow a = 6\alpha$

$b = 5\alpha$

$c = 4\alpha$

$\Rightarrow$ semiperimeter $s = \dfrac{6\alpha + 5\alpha + 4\alpha}{2}$

$= \dfrac{15\alpha}{2}$ — (1)

$\Rightarrow$ By Heron's formula

$ar(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}$

$\Rightarrow 30\alpha = \sqrt{\frac{15\alpha}{2}\left(\frac{3\alpha}{2}\right)\left(\frac{5\alpha}{2}\right)\left(\frac{7\alpha}{2}\right)}$

$\Rightarrow 30\alpha = \dfrac{15\alpha^2}{4}\sqrt{7}$

$\Rightarrow 120\cancel{\alpha} = 15\alpha^{\cancel{2}}\sqrt{7}$

$\Rightarrow \cancel{120}^{8} = \cancel{15}_{1}\alpha\sqrt{7}$

$\Rightarrow \dfrac{8}{\sqrt{7}} = \alpha$

$\Rightarrow \dfrac{8\sqrt{7}}{7} = \alpha$ — (2)

Substituting eq (2) in (1)

$\Rightarrow s = \dfrac{15}{\cancel{2}_1} \times \dfrac{\cancel{8}^{4}\sqrt{7}}{7} = \boxed{\dfrac{60\sqrt{7}}{7}} / \boxed{\dfrac{60}{\sqrt{7}}}$

18.7 Given $x^2+4y^2+16z^2=48$

$\Rightarrow (x)^2+(2y)^2+(4z)^2=48$ — ①

$xy+4yz+2zx=24$

$\Rightarrow 4xy+16yz+8zx=96$ — ②

$\Rightarrow 2xy+8yz+4zx=48$ — ③

eq ① − eq ③ = 0

$\Rightarrow x^2+4y^2+16z^2 = 2xy+8yz+4zx$

$\Rightarrow \frac{1}{2}\left((x-2y)^2+(2y-4z)^2+(4z-x)^2\right)=0$

$\Rightarrow \left((x-2y)^2+(2y-4z)^2+(4z-x)^2\right)=0$

~~$\Rightarrow x^2-4yx+4y^2+4y^2-16yz+16yz^2+16z^2-8zx+x^2=0$~~

~~$\Rightarrow 2x^2+8y^2+32z^2-(4xy+16yz-8zx)=0$~~

$\Rightarrow (x-2y)^2=(2y-4z)^2=(4z-x)^2=0$

$\Rightarrow x-2y=0$

$2y-4z=0$

$4z-x=0$

$\Rightarrow x=2y=4z$ — ④

Substituting eq ④ in ①

we have

$(x)^2+(x)^2+(x)^2=48$

$\Rightarrow 3(x)^2=48$

$\Rightarrow x^2=16$

$\Rightarrow x=4$

$\Rightarrow y=x/2=4/2=2$

$\Rightarrow z=x/4=4/4=1$

$\Rightarrow x^2+y^2+z^2=(4)^2+(2)^2+(1)^2$

$=16+4+1$

$=\boxed{21}$

19.7 $x^4 + ax^2 + bx - c = 0$ has roots 1, 2, 3.

Let the fourth root be α.

Sum of roots $= \frac{0}{1} = 0$

$\Rightarrow 1 + 2 + 3 + \alpha = 0$

$\Rightarrow 6 + \alpha = 0$

$\Rightarrow \alpha = -6$

Product of roots $= -\frac{c}{1}$

$\Rightarrow (1)(2)(3)(\alpha) = -c$

$\Rightarrow 6(-6) = -c$

$\Rightarrow -36 = -c$

$\Rightarrow c = \boxed{36}$ //

21.> The possible factorisations of 108 are

i.> $108 \times 1 \times 1$

ii.> $54 \times 2 \times 1$

iii.> $36 \times 3 \times 1$

iv.> $27 \times 4 \times 1$

v.> $18 \times 6 \times 1$

vi.> $12 \times 9 \times 1$

vii.> $27 \times 2 \times 2$

viii.> $18 \times 3 \times 2$

ix.> $9 \times 6 \times 2$

x.> $12 \times 3 \times 3$

xi.> $9 \times 4 \times 3$

xii.> $6 \times 6 \times 3$

The number of ordered triplets of each factorisation $= 3!$

$\therefore$ Total number of ordered triplets $= 3! \times 12 = 12 \times 6 = \boxed{72}$

22.>

23.7 $7x^2 - 44x + 12 = p^n$

$\Rightarrow 7x^2 - 42x - 2x + 12 = p^n$

$\Rightarrow 7x(x - 6) - 2(x - 6) = p^n$

$\Rightarrow (7x - 2)(x - 6) = p^n$

$\Rightarrow (7x-2)$ & $(x-6)$ are powers of prime p.

Observations –

- $x > 6$ as otherwise it cannot be a power of p as $x-6$ would be 0 or negative.
- x is odd as otherwise the product won't be of the form $p^i \times p^{n-i}$

From observation ① and trial & error we find that for $x = 7$, $p = 47$ is the smallest value of p.

And from observation ② and the fact that x cannot be a ~~factor of~~ multiple of 3 as then p should be 3 as 3 would be a factor.

$\therefore$ – The next value that can equal $x = 11$.

But $7x - 2 = 75$ which is not a prime

– The next value is 13.

But $7x - 2 = 91 - 2 = 89$, $x - 6 = 7$ which is 2 primes and hence not possible.

– The next value is 15.

But $7x - 2 = 105 - 2 = 103$, $15 - 6 = 9$ which is not a prime and the value of $7x - 2$ crosses 100 and since we know that the value of the prime is a 2-digit number through the pattern of the OMR sheet x cannot cross the value of 15.

$\therefore$ The largest value of p satisfying the equation is $\boxed{47}$ which is also the smallest value of p satisfying the equation.